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Stable Extensions of Constrained Optimization Problems

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One extension of a constrained optimization problem defined on a uniform space is proposed. It is shown that, if the cost function is uniformly continuous and the multifunction forming the constraint is both upper and lower Hausdorff semicontinuous, the extension is stable in the sense that the infimum depends continuously on a parameter and the solution-set multifunction is upper semicontinuous. These results are interpreted for the original problem, giving some assertions without any compactness requirements. Besides, it is demonstrated that the extension, being constructed by means of the precompact modification of the original uniformity, cannot be constructed generally by any other uniformity.

1. INTRODUCTION AND CLASSICAL RESULTS

We will deal with the family $\{P_y\}_{y \in Y}$ of constrained minimization problems

$$P_y: \text{minimize } f(x, y) \quad \text{subject to } x \in G(y),$$

where $f: X \times Y \to \bar{R}$ is a cost function, X is a set (which will be endowed by some topology or uniformity), Y is a topological space, $\bar{R} = R \cup \{\pm\infty\}$ is the standard two-point compactification of the real line, and $G: Y \to 2^X$ is a set-valued mapping from Y to X (we will say briefly a multifunction). Such a problem has been treated many times, see, e.g., [1–3, 5, 6], but all of these results are derived under some more or less strong compactness conditions. The first aim of the paper is to state some results without any compactness condition which will be replaced by a requirement of a certain uniform continuity of the data. The results will be proved by using a suitable extension of the problem. It will be performed in Section 2. The second aim, treated in Section 3, is to show that there does not exist any other extension having the properties of the extension used in Section 2.

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We briefly recall some standard definitions. Let σ be a topology on X. We say that the multifunction G is upper semicontinuous with respect to σ (briefly σ-u.s.c.) at $y \in Y$ if for every σ-neighbourhood A of $G(y)$ there is a neighbourhood B of y such that $G(\bar{y}) \subset A$ for every $\bar{y} \in B$. To make our expressions clear and short, we will often put the respective structures as a prefix (e.g., a σ-neighbourhood, etc.). The topology on Y will not be specified. Of course, if G is σ-u.s.c. at every $y \in Y$, we will say that G is σ-u.s.c. (and similarly for other types of semicontinuity). The multifunction G is called lower semicontinuous with respect to σ (briefly σ-l.s.c.) at $y \in Y$ if for every σ-open $A \subset X$ with $A \cap G(y) \neq \emptyset$ there is a neighbourhood B of y such that $A \cap G(\bar{y}) \neq \emptyset$ for every $\bar{y} \in B$. We introduce the marginal function $m\colon Y \to \bar{R}$ defined by $m(y) = \inf_{x \in G(y)} f(x, y)$; i.e., $m(y)$ is the infimum of P_y. The marginal multifunction $M\colon Y \to 2^X$ is defined by $M(y) = \{x \in G(y); f(x, y) = m(y)\}$; i.e., $M(y)$ is the set of the minimizers of P_y. We present the well-known classical results (see, e.g., J.-P. Aubin and I. Ekeland [1, Chap. 3, Sect. 1, Proposition 23]):

PROPOSITION 1.1. *Let f be continuous, G σ-*u.s.c. *and σ-*l.s.c. *with compact values (i.e., $G(y)$ is σ-compact for every $y \in Y$). Then m is continuous and M is σ-*u.s.c.

When f does not depend on y, we obtain the classical theorem of C. Berge [3, VI.3]. Of course, the compactness of the values of G is considerably restrictive. Yet this requirement can be somewhat weakened. For the case that f is independent on y, the topology on Y fulfills the first countability axiom, and X is a complete metric space, it was done by E. Bednarczuk [2, Theorem 8], who supposed f to be continuous, G σ-u.s.c. and σ-l.s.c., and, for every $y \in Y$: $\mathrm{Act}_{\mathrm{I}} G(y) \subset G(y)$ and $\psi(\bigcup_{\bar{y} \in B_n} G(y) \setminus G(\bar{y})) \to 0$ for $n \to +\infty$, where $\mathrm{Act}_{\mathrm{I}}$ is the set of so-called innerly active points (see [2]), ψ is the measure of noncompactness, and $\{B_n\}_{n \in N}$ is a countable base of the neighbourhood filter of y, N being the set of all natural numbers. We may observe that the assumption of compactness of $G(y)$ has been replaced by the requirement that, roughly speaking, the part of $G(y)$, exhibiting changes when y moves, is "nearly compact." For the changes that $G(y)$ increases it is ensured just by the assumption that the measure of noncompactness is "small" near y, while for the changes when $G(y)$ increases it follows just from the upper semicontinuity of G which ensures the compactness of the so-called active boundary of $G(y)$; see S. Dolecki and S. Rolewicz [6]. It should be emphasized that generally these assumptions cannot be weakened.

2. A Stable Extension of the Problem

From now on we suppose X to be endowed by a uniformity $\mathscr{U}$. We recall some standard definition from the uniform space theory; see, e.g., [4, 7]. A uniformity $\mathscr{U}$ on X is a filter on $X \times X$ with the following properties: $\forall U \in \mathscr{U}$: $\Delta \subset U$, $U^{-1} \in \mathscr{U}$, and $\exists V \in \mathscr{U}$: $V \circ V \subset U$; where $\Delta = \{(x, x); x \in X\}$ is the identity relation on X, $U^{-1} = \{(x_1, x_2); x_2 U x_1\}$ is the inverse relation to the binary relation U (we will use the infix notation that is standard for binary relations, e.g., $x_1 U x_2$ means that $(x_1, x_2) \in U$), and $V \circ V = \{(x_1, x_2); \exists x_3 \colon x_1 V x_3, x_3 V x_2\}$ is the composite relation. A filter $\mathscr{F}$ on a set Z is a nonempty subset of 2^Z such that $\emptyset \notin \mathscr{F}$, $A \in \mathscr{F}$ whenever there is $A_0 \in \mathscr{F}$ such that $A \supset A_0$, and $A_1 \cap A_2$ whenever $A_1, A_2 \in \mathscr{F}$. For simplicity, we will suppose $\mathscr{U}$ as a Hausdorff uniformity, i.e., if $x_1 \neq x_2$, then $x_1 U x_2$ is not true for some $U \in \mathscr{U}$. In applications the most frequent case is that (X, d) is a metric space, and $\mathscr{U}$ is induced by the metric d (we will write $\mathscr{U} = \mathscr{U}_d$): $\mathscr{U}_d = \{U \subset X \times X; \exists \varepsilon > 0\colon d(x_1, x_2) \leqslant \varepsilon \Rightarrow x_1 U x_2\}$. The uniformity $\mathscr{U}$ induces a topology on X, denoted by $\tau(\mathscr{U})$, by declaring $\bigcap_{U \in \mathscr{U}} U(A)$ as the $\tau(\mathscr{U})$-closure of a set $A \subset X$; where $U(A) = \{x \in X; \exists \bar{x} \in A\colon x U \bar{x}\}$ is a so-called uniform neighbourhood of A. If $A = \{x\}$, we will write briefly $U(x)$ instead of $U(\{x\})$. The topology $\tau(\mathscr{U})$ is completely regular (i.e., any closed set and a point disjoint with it can be separated by a continuous function) and, conversely, any completely regular topology can be induced by some uniformity. Of course, the topology induced by $\mathscr{U}_d$ coincides with the standard topology of the metric space (X, d).

The multifunction G is said to be upper Hausdorff semicontinuous with respect to the uniformity $\mathscr{U}$ (briefly $\mathscr{U}$-u.H.s.c.) at $y \in Y$ if for every $U \in \mathscr{U}$ there is a neighbourhood B of y such that $G(\bar{y}) \subset U(G(y))$ for every $\bar{y} \in B$. The multifunction G is called lower Hausdorff semicontinuous with respect to $\mathscr{U}$ (briefly $\mathscr{U}$-l.H.s.c.) at $y \in Y$ if for every $U \in \mathscr{U}$ there is a neighbourhood B of y such that $G(y) \subset U(G(\bar{y}))$ for every $\bar{y} \in B$. In the metric case we obtain naturally the standard definitions; for the relations between various types of semicontinuities we refer to S. Dolecki [5]. Let us remark that G is simultaneously $\mathscr{U}$-u.H.s.c. and $\mathscr{U}$-l.H.s.c. if and only if G, considered as a single-valued mapping $Y \to 2^X$, is continuous with respect to $\tau(\mathscr{U}^{\mathrm{H}})$, where $\mathscr{U}^{\mathrm{H}} = \{W \subset 2^X \times 2^X; \exists U \in \mathscr{U}\colon A_1, A_2 \subset X, A_1 \subset U(A_2), A_2 \subset U(A_1) \Rightarrow A_1 W A_2\}$ is the so-called hyper-space uniformity on 2^X induced by the uniformity $\mathscr{U}$ on X; see [7]. In the metric case, $\mathscr{U}_d^{\mathrm{H}}$ is just the uniformity induced by the well-known Hausdorff pseudo-metric d^{H} defined by $d^{\mathrm{H}}(A_1, A_2) = \max(\sup_{x \in A_1} \operatorname{dist}(x, A_2), \sup_{x \in A_2} \operatorname{dist}(x, A_1))$, where $\operatorname{dist}(x, A) = \inf d(x, A)$ is the distance of x from A with the convention that $\inf \emptyset = +\infty$.

For two uniformities $\mathscr{U}_1$, $\mathscr{U}_2$ on X we say that $\mathscr{U}_1$ is coarser than $\mathscr{U}_2$, or $\mathscr{U}_2$ is finer than $\mathscr{U}_1$ if $\mathscr{U}_1 \subset \mathscr{U}_2$. Also, $\mathscr{U}_1$ is finer than $\mathscr{U}_2$ if and only if the identity on X is $(\mathscr{U}_1, \mathscr{U}_2)$-uniformly continuous; we say that a mapping φ is $(\mathscr{U}_1, \mathscr{U}_2)$-uniformly continuous if $\forall U \in \mathscr{U}_2\ \exists V \in \mathscr{U}_1$: $x_1 V x_2 \Rightarrow \varphi(x_1)\, U \varphi(x_2)$. We denote by $\mathscr{U}^*$ the coarsest uniformity $\mathscr{V}$ on X that makes $(\mathscr{V}, \mathscr{R})$-uniformly continuous all the $(\mathscr{U}, \mathscr{R})$-uniformly continuous functions from X to $\bar{R}$, where $\mathscr{R}$ denotes the uniformity on $\bar{R}$ inducing the standard compact topology of $\bar{R}$ (recall that on a compact space there exists exactly one uniformity inducing its topology). The uniformity $\mathscr{U}^*$, called the precompact modification of $\mathscr{U}$, is the finest among all the precompact uniformities which are coarser than $\mathscr{U}$. Recall that a uniformity $\mathscr{V}$ is called to be precompact if $\forall V \in \mathscr{V}\ \exists x_1, ..., x_n \in X$: $\bigcup_{i=1}^n V(x_i) = X$. Clearly, $\mathscr{U}^*$ is coarser than $\mathscr{U}$, and $\mathscr{U}$ is precompact iff $\mathscr{U}^* = \mathscr{U}$. Alternatively we may define $\mathscr{U}^*$ by (cf. [7, Theorem 12.3])

$$\mathscr{U}^* = \left\{ U \subset X \times X; \exists A_1, ..., A_n \subset X\ \exists V \in \mathscr{U}: \bigcup_{i=1}^n A_i = X, \right.$$
$$\left. U \supset \bigcup_{i=1}^n V(A_i) \times V(A_i) \right\}.$$

For a uniformity $\mathscr{V}$ on X, a filter $\mathscr{F}$ on X is said to be $\mathscr{V}$-Cauchy if $\forall V \in \mathscr{V}\ \exists A \in \mathscr{F}$: $A \times A \subset V$. The uniform space $(X, \mathscr{V})$ is called to be complete if every $\mathscr{V}$-Cauchy filter on X is $\tau(\mathscr{V})$-convergent; we say that $\mathscr{F}$ is τ-convergent if there exists $x \in X$ such that every τ-neighbourhood of x belongs to $\mathscr{F}$. It is well known that, if $(X, \mathscr{V})$ is a Hausdorff uniform space, there exists a uniquely determined (up to a homeomorphism) complete Hausdorff uniform space $(X_c^{\mathscr{V}}, \mathscr{V}_c)$, called the completion of $(X, \mathscr{V})$, such that X is a $\tau(\mathscr{V}_c)$-dense subset of $X_c^{\mathscr{V}}$ and the trace on $X \times X$ of $\mathscr{V}_c$ is just $\mathscr{V}$. We use $\mathscr{V}$ as a superscript for the set $X_c^{\mathscr{V}}$ to indicate its dependence on $\mathscr{V}$. The completion is $\tau(\mathscr{V}_c)$-compact if and only if $\mathscr{V}$ is precompact.

Let us mention that $\mathscr{U}_d^*$ is metrizable if and only if $\mathscr{U}_d$ is precompact, which follows from the facts that $\mathscr{U}_d^*$ and $\mathscr{U}_d$ induce the same proximity (for the definition of a proximity we refer to [4, 7]); $\mathscr{U}_d^*$, being precompact, is the coarsest uniformity inducing this proximity; and simultaneously $\mathscr{U}_d^*$ is the finest uniformity inducing this proximity because it is assumed metrizable [7, Theorem 12.19]. Thus we can see that the uniformity $\mathscr{U}_d^*$ hardly can be metrizable and therefore it may seem somewhat "etheric" (except the rather trivial case when $\mathscr{U}_d$ is precompact), but it should be pointed out that we will formulate our assumptions on the data f and G in terms of the original uniformity $\mathscr{U}$ only.

We assume

$$\forall y \in Y\ \forall V \in \mathscr{R}\ \exists U \in \mathscr{U}\ \forall x_1, x_2 \in X: x_1 U x_2 \Rightarrow f(x_1, y)\, V f(x_2, y). \tag{2.1}$$

In other words, $f(\cdot, y): X \to \bar{R}$ is to be $(\mathscr{U}, \mathscr{R})$-uniformly continuous for every $y \in Y$. Note that the trace on R of $\mathscr{R}$ is coarser than the usual additive uniformity $\mathscr{R}_0 = \{V \subset R \times R;\ \exists \varepsilon > 0\colon |a_1 - a_2| \leqslant \varepsilon \Rightarrow a_1 V a_2\}$ on R; e.g., the function $a \mapsto a^2$, which is not $(\mathscr{R}_0, \mathscr{R}_0)$-uniformly continuous, is $(\mathscr{R}_0, \mathscr{R})$-uniformly continuous.

Under the assumption (2.1) we may extend the problem to the completion of $(X, \mathscr{U}^*)$. By the definition of $\mathscr{U}^*$, $f(\cdot, y)$ is also $(\mathscr{U}^*, \mathscr{R})$-uniformly continuous. Using the fact that the uniform space $(\bar{R}, \mathscr{R})$ is complete, we may extend $f(\cdot, y)$ by continuity (see [4, Theorem 6.2.7]) to $\bar{f}(\cdot, y)\colon \bar{X} \to \bar{R}$, where we use the abbreviation $\bar{X} = X_c^{\mathscr{U}^*}$. Thus we obtain the function $\bar{f}\colon \bar{X} \times Y \to \bar{R}$ defined by

$$\bar{f}(x, y) = \lim_{\tilde{x} \to x,\, \tilde{x} \in X} f(\tilde{x}, y), \tag{2.2}$$

where $\tilde{x} \to x$ means, of course, the convergence in the topology $\tau(\mathscr{U}_c^*)$; of course, $\mathscr{U}_c^*$ means $(\mathscr{U}^*)_c$. Furthermore, we define the multifunction $\bar{G}\colon Y \to 2^{\bar{X}}$ by

$$\bar{G}(y) = \mathrm{cl}_{\bar{X}}\, G(y), \tag{2.3}$$

where $\mathrm{cl}_{\bar{X}}$ means the $\tau(\mathscr{U}_c^*)$-closure. Thus we get the extended problem

$$\bar{P}_y\colon \text{minimize } \bar{f}(x, y)$$
$$\text{subject to } x \in \bar{G}(y).$$

We define naturally the marginal function $\bar{m}\colon Y \to \bar{R}$ by $\bar{m}(y) = \inf \bar{f}(\bar{G}(y), y)$ (i.e., $\bar{m}(y)$ is the infimum of $\bar{P}_y$), and the marginal multifunction $\bar{M}\colon Y \to 2^{\bar{X}}$ by $\bar{M}(y) = \{x \in \bar{G}(y);\ \bar{f}(x, y) = \bar{m}(y)\}$. The assumption (2.1) guarantees the continuity of $\bar{f}(\cdot, y)$, thus also the equality $\bar{m}(y) = m(y)$ for every $y \in Y$. We observe that the problem $\bar{P}_y$ is a continuous extension of the original problem P_y and, moreover, has a compact domain $\bar{X}$. We remark that the space $\bar{X}$ with the compact topology $\tau(\mathscr{U}_c^*)$ is called the Samuel compactification of the uniform space $(X, \mathscr{U})$. Also, it is homeomorphic to the Smirnov compactification of X with respect to the proximity induced by $\mathscr{U}$ (for details see, e.g., [4, 7]).

Now we impose further assumptions on the original data f and G to ensure stable behaviour of the extended problem

$$\forall y \in Y\ \forall V \in \mathscr{R}\ \exists \text{ a neighbourhood } B \text{ of } y$$
$$\forall \bar{y} \in B\ \forall x \in X\colon f(x, \bar{y})\ V f(x, y). \tag{2.4}$$

In other words, the family of functions $\{f(x, \cdot)\colon Y \to \bar{R}\}_{x \in X}$ is to be equicontinuous with respect to the uniformity $\mathscr{R}$. As to the multifunction G, we assume

$$G \text{ is } \mathscr{U}\text{-u.H.s.c. and } \mathscr{U}\text{-l.H.s.c.} \tag{2.5}$$

PROPOSITION 2.1. *If f fulfills* (2.1) *and* (2.4), *then $\bar{f}$ defined by* (2.2) *is continuous. If G fulfills* (2.5), *then $\bar{G}$ defined by* (2.3) *is* $\mathscr{U}_c^*$-u.H.s.c. *and* $\mathscr{U}_c^*$-l.H.s.c.

Proof. Let $(x, y) \in \bar{X} \times Y$ and $W \in \mathscr{R}$ be given. Take a symmetric $V \in \mathscr{R}$ such that $V \circ V \circ V \circ V \subset W$. Due to (2.1), $f(\cdot, y)$ is $(\mathscr{U}^*, \mathscr{R})$-uniformly continuous, hence the extension $\bar{f}(\cdot, y)$ is $(\mathscr{U}_c^*, \mathscr{R})$-uniformly continuous, and thus there is $U \in \mathscr{U}_c^*$ such that

$$x_1 U x_2 \Rightarrow \bar{f}(x_1, y)\; V \bar{f}(x_2, y). \tag{2.6}$$

In view of (2.4) we can take some neighbourhood B of y such that

$$\forall \bar{x} \in X\; \forall y_1 \in B: f(\bar{x}, y_1)\; V f(\bar{x}, y). \tag{2.7}$$

Let $(x_1, y_1) \in U(x) \times B$. Then, for every $\tilde{x} \in U(x_1) \cap X$, we have the estimate $f(\tilde{x}, y_1)(V \circ V \circ V)\bar{f}(x, y)$ because $f(\tilde{x}, y_1)\, V f(\tilde{x}, y)$, $f(\tilde{x}, y)\, V\bar{f}(x_1, y)$, and $\bar{f}(x_1, y)\, V\bar{f}(x, y)$; obviously, (2.7), (2.6), and again (2.6) have been employed, respectively. In view of (2.2) we have got $\bar{f}(x_1, y_1) \in \operatorname{cl}_{\bar{R}} V(V(V(\bar{f}(x, y)))) \subset W(\bar{f}(x, y))$. To summarize, for every $(x, y) \in \bar{X} \times Y$ and $W \in \mathscr{R}$ we have found a neighbourhood $U(x) \times B$ of (x, y) such that $\bar{f}(x_1, y_1)\, W\bar{f}(x, y)$ whenever $(x_1, y_1) \in U(x) \times B$, that is just the continuity of $\bar{f}$.

The continuity of $\bar{G}$ follows from (2.5) by the facts that $\mathscr{U}^*$ is coarser than $\mathscr{U}$ (hence G is also $\mathscr{U}^*$-u.H.s.c. and $\mathscr{U}^*$-l.H.s.c.) and that the closure operator preserves the Hausdorff semicontinuity of the multifunctions. Indeed, for every $U \in \mathscr{U}_c^*$ we can take $V \in \mathscr{U}_c^*$ and a neighbourhood B of y such that $V \circ V \subset U$ and, for every $\bar{y} \in B$, $G(\bar{y}) \subset W(G(y))$ with $W = V \cap (X \times X) \in \mathscr{U}^* \subset \mathscr{U}$. Then we have the estimate $\bar{G}(\bar{y}) \subset V(G(\bar{y})) \subset V(W(G(y))) \subset U(\bar{G}(y))$, which shows that $\bar{G}$ is $\mathscr{U}_c^*$-u.H.s.c. The lower Hausdorff semicontinuity can be treated analogously. ∎

Remark 2.1. Note that (2.1) and (2.4) are not only sufficient conditions for $\bar{f}$ to be continuous, but also necessary. Indeed, if $\bar{f}$ is continuous, also $\bar{f}(\cdot, y)$ is continuous for every $y \in Y$, and then (2.1) follows by the compactness of $\bar{X}$. Now suppose that (2.4) does not hold, i.e., we have some $y \in Y$ and $V \in \mathscr{R}$ such that for every neighbourhood B of y there is $y_B \in B$ and $x_B \in X$ such that $f(x_B, y_B) \in X \backslash V(f(x_B, y))$. Due to the compactness of $\bar{X}$ the net $\{x_B\}_{B \in \mathscr{B}}$, $\mathscr{B}$ being the neighbourhood filter of y, has some cluster point $x \in \bar{X}$. Take a symmetric $W \in \mathscr{R}$ with $W \circ W \subset V$. Since $\bar{f}(\cdot, y)$ is continuous, for a sufficiently small B we have $f(x_B, y) \in W(\bar{f}(x, y))$. Thus we have got the situation: (x, y) is a cluster point of the net $\{(x_B, y_B)\}_{B \in \mathscr{B}}$ in $\bar{X} \times Y$, but $f(x_B, y_B) \in X \backslash V(f(x_B, y)) \subset X \backslash W(\bar{f}(x, y))$ for all B small enough, thus $\bar{f}$ cannot be continuous at the point (x, y), which is the contradiction showing that (2.4) must be valid provided $\bar{f}$ should be continuous.

Remark 2.2. Note that, while the topology of X must be normal for the closure operator to preserve the upper semicontinuity (see [5]), the normality is not necessary to preserve the upper Hausdorff semicontinuity.

THEOREM 2.1. *Let f fulfill* (2.1) *and* (2.4), *and G fulfill* (2.5). *Then $\bar{m}$ is continuous and $\bar{M}$ is* $\tau(\mathscr{U}_c^*)$-u.s.c.

Proof. The assertion can be obtained immediately by applying Proposition 1.1 to the extended problem $\bar{P}_y$ and using Proposition 2.1 together with the compactness of $\bar{X}$ (which guarantees the compactness of $\bar{G}(y)$ for every $y \in Y$) and the equivalence of the mere semicontinuities with the Hausdorff ones. ∎

The following corollary represents a certain analogy with the classical assertion of Proposition 1.1 concerning the marginal function m.

COROLLARY 2.1. *Under the assumptions* (2.1), (2.4), *and* (2.5), *the marginal function m of the original problem is continuous.*

Proof. It follows immediately from Theorem 2.1 and from the equality $\bar{m} = m$ which is guaranteed by (2.1). ∎

We see that, thanks to the equality $\bar{m} = m$, the interpretation of the stability of the infimum of the extended problem is straightforward, indeed. On the contrary, an interpretation of the stability of the marginal multifunction $\bar{M}$ is somewhat more complicated.

The simplest, though rather formal interpretation of this stability can be performed be declaring the elements of $\bar{M}(y)$ as the generalized solutions of the original problem P_y; cf. also [9, 10]. Roughly speaking, the generalized solutions represent certain analogy with the classical notion of (classes of some) minimizing sequences of P_y, or more exactly, minimizing nets. To make the nature of the generalized solutions clear, we state some more or less usual definitions. A filter $\mathscr{F}$ on X is called $\mathscr{U}$-round if $\forall A \in \mathscr{F}\ \exists U \in \mathscr{U}\ \exists A_0 \in \mathscr{F}$: $U(A_0) \subset A$; cf. [4, 7] (the property "to be round" is, in fact, a proximal property, but we consider it as a uniform property for the sake of simplicity). If, in addition, there is no $\mathscr{U}$-round filter $\tilde{\mathscr{F}}$ with $\tilde{\mathscr{F}} \neq \mathscr{F}$ and $\tilde{\mathscr{F}} \supset \mathscr{F}$, $\mathscr{F}$ is said to be a maximal $\mathscr{U}$-round filter. We say that $\mathscr{F}$ is y-feasible if $\forall A \in \mathscr{F}: A \cap G(y) \neq \varnothing$.

PROPOSITION 2.2. *There is a one-to-one correspondence between the elements of $\bar{M}(y)$, declared as the generalized solutions of P_y, and the y-feasible maximal $\mathscr{U}$-round filters on X with the property*: $\limsup_{\mathscr{F}} f(\cdot, y) \leqslant \limsup_{\tilde{\mathscr{F}}} f(\cdot, y)$ *for every y-feasible maximal $\mathscr{U}$-round filter $\tilde{\mathscr{F}}$ on X; where, for $\varphi: X \to \bar{R}$,* $\limsup_{\mathscr{F}} \varphi$ *means naturally* $\inf_{A \in \mathscr{F}} \sup_{x \in A} \varphi(x)$.

Proof. By means of the mapping $x \mapsto \mathcal{N}(x)$, where $\mathcal{N}(x) = \{A \cap X;\ A$ is a $\tau(\mathcal{U}_c^*)$-neighbourhood of $x\}$, the elements of X can be identified by a one-to-one manner with all maximal $\mathcal{U}$-round filters on X; see [4, Theorems 6.4.8 and 6.3.12]. The y-feasible maximal $\mathcal{U}$-round filters can then be identified just with the elements of $\bar{G}(y)$. Finally, the assertion follows by the facts that $\bar{f}(x, y) = \limsup_{\mathcal{N}(x)} f(\cdot, y)$, and $x \in \bar{M}(y)$ iff $x \in \bar{G}(y)$ and $\bar{f}(x, y) \leqslant \bar{f}(\tilde{x}, y)$ for every $\tilde{x} \in \bar{G}(y)$. ∎

In such interpretation, Theorem 2.1 asserts that the set of the generalized solutions is stable (u.s.c.), which makes the notion of the generalized solutions sensible. It should be emphasized that, under the assumptions of Theorem 2.1, there is no stability of the set $M(y)$ of the classical solutions which may be even empty.

In the metric case (i.e., $\mathcal{U} = \mathcal{U}_d$, d being a metric on X) the stability of the marginal multifunction $\bar{M}$ can be employed still in another, less formal manner; namely to establish certain stability for the minimizing sequences of the original problem. As usual (see, e.g., E. Polak and Y. Y. Wardi [8]) we say that a sequence $s = \{x_n\}_{n \in N}$, $x_n \in X$ and N the set of all natural numbers, is feasible for P_y (briefly y-feasible) if $\lim_{n \to \infty} \operatorname{dist}(x_n, G(y)) = 0$. This sequence is called y-minimizing if it is y-feasible and $\limsup_{n \to \infty} f(x_n, y) \leqslant \limsup_{n \to \infty} f(\bar{x}_n, y)$ for every y-feasible sequence $\{\bar{x}_n\}_{n \in N}$. The feasible sequences are sometimes also called eventually feasible or asymptotically admissible. From Theorem 2.1 we get the following corollary:

COROLLARY 2.2. *Let $\mathcal{U} = \mathcal{U}_d$, (2.1), (2.4), and (2.5) be valid, $\{y_n\}_{n \in N}$ be some sequence in Y converging to y, and, for every $n \in N$, $s^n = \{x_m^n\}_{m \in N}$ be a y_n-minimizing sequence. Then there exists a function $\kappa\colon N \to N$ such that every sequence $s = \{x_{k_n}^n\}_{n \in N}$ with $k_n \geqslant \kappa(n)$ is y-minimizing.*

Proof. From [10, Theorem 3 with $X = Y$, $\mathcal{U}_X = \mathcal{U}_Y = \mathcal{U}_d$, F the identity on X, and $C = G(y)$] we can see that the filter $\mathcal{M}(y) = \{A \cap X;\ A$ is a $\tau(\mathcal{U}_c^*)$-neighbourhood of $\bar{M}(y)\}$ has got a countable base, e.g., $\{A_j\}_{j \in N}$ with $A_j = \{x \in X;\ \operatorname{dist}(x, G(y)) \leqslant 1/j,\ f(x, y) \leqslant m(y) + 1/j\}$. By [11, Proposition 1.2d] the sequence $s = \{x_n\}_{n \in N}$ is y-minimizing if and only if the corresponding sequential filter $\mathcal{S}(s) = \{A \subset X;\ \exists n_0 \in N\ \forall n \geqslant n_0\colon x_n \in A\}$ on X is finer than $\mathcal{M}(y)$. Now we employ Theorem 2.1 and [11, Lemma 5.1] to show that the sequence of the filters $\mathcal{M}(y_n)$ converges to $\mathcal{M}(y)$ in the sense: $\forall A \in \mathcal{M}(y)\ \exists n_0 \in N\ \forall n \geqslant n_0\colon A \in \mathcal{M}(y_n)$. The existence of a function κ with the desired properties then follows directly from [11, Proposition 2.3]. ∎

Remark 2.3. We observe that, on one hand, the uniformity $\mathcal{U}$ must be fine enough to ensure (2.1), and, on the other hand, it must be sufficiently coarse to guarantee (2.5). In other words, for the existence of a suitable uniformity $\mathscr{U}$ the following is necessary: the worse the continuity properties of f, the better the continuity properties of G (and vice versa).

Remark 2.4. It is evident that all the assertions remain valid if we replace $\mathscr{U}$ by $\mathscr{U}^*$ in the assumptions (2.1) and (2.5), thus obtaining

$$f(\cdot, y) \text{ is } (\mathscr{U}^*, \mathscr{R})\text{-uniformly continuous for every } y \in Y, \qquad (2.1)^*$$

$$G \text{ is } \mathscr{U}^*\text{-u.H.s.c. and } \mathscr{U}^*\text{-l.H.s.c.} \qquad (2.5)^*$$

While $(2.1)^*$ is equivalent to (2.1) by the very definition of $\mathscr{U}^*$, $(2.5)^*$ is weaker than (2.5) provided $\mathscr{U}^* \neq \mathscr{U}$, i.e., $\mathscr{U}$ is not precompact. Indeed, (2.5) or $(2.5)^*$ is equivalent to the continuity of $G: Y \to 2^X$ (considered as a single-valued mapping) with respect to the topology $\tau(\mathscr{U}^{\mathrm{H}})$ or $\tau((\mathscr{U}^*)^{\mathrm{H}})$ on 2^X, respectively. However, these topologies actually differ from each other provided $\mathscr{U} \neq \mathscr{U}^*$; see [7, Corollary 15.2]. Hence the modification of Theorem 2.1 obtained when (2.5) is replaced by $(2.5)^*$ represents a stronger result indeed, though the condition $(2.5)^*$ using the uniformity $\mathscr{U}^*$ is rather ineffective.

The preceding remarks suggest the idea to choose various uniformities for $\mathscr{U}$. Indeed, using Corollary 2.1 with a special choice of $\mathscr{U}$, we can obtain directly a generalization (at least if f does not depend on y) of the classical result in Proposition 1.1 concerning the marginal function m:

COROLLARY 2.3. *Let (X, σ) be a completely regular topological space, $f(\cdot, y): X \to \bar{R}$ be continuous for every $y \in Y$, f fulfill* (2.4), *and G be σ-u.s.c. and σ-l.s.c. Then the marginal function m is continuous.*

Proof. We take $\mathscr{U} = \mathscr{U}_\sigma^*$, where $\mathscr{U}_\sigma$ denotes the finest uniformity on X inducing the topology σ. Then $\mathscr{U}_\sigma^*$ is the so-called Stone–Čech uniformity of the completely regular space (X, σ), and $\bar{X}$ is just the well-known Stone–Čech compactification of this space. The uniformity $\mathscr{U}_\sigma^*$ can be projectively generated by the family of all σ-continuous functions from X to $[0, 1]$. As $(\bar{R}, \mathscr{R})$ is uniformly homeomorphic to the interval $[0, 1]$, every continuous function from X to $\bar{R}$ is $(\mathscr{U}_\sigma^*, \mathscr{R})$-uniformly continuous as well, thus f satisfies (2.1).

Since G is σ-u.s.c., it is evidently also $\mathscr{U}_\sigma$-u.H.s.c. because $\mathscr{U}_\sigma$ induces σ, and therefore it is $\mathscr{U}_\sigma^*$-u.H.s.c. as well because $\mathscr{U}_\sigma^* \subset \mathscr{U}_\sigma$. As for the lower semicontinuity, we must employ the precompactness of $\mathscr{U}_\sigma^*$. Let $U \in \mathscr{U}_\sigma^*$ and $y \in Y$ be given. Take a symmetric $V \in \mathscr{U}_\sigma^*$ such that $V \circ V \subset U$. Since $\mathscr{U}_\sigma^*$ is precompact, there is a finite set $\{x_k\}_{k=1}^n \subset G(y)$ such that $\bigcup_{k=1}^n V(x_k) \subset G(y)$. As G is assumed σ-l.s.c., we can take a neighbourhood B of y such that $G(\bar{y}) \cap V(x_k)$ is nonempty for every $\bar{y} \in B$ and $k = 1, ..., n$. In other words, $x_k \in V(G(\bar{y}))$. Now we get the estimate: $G(y) \subset \bigcup_{k=1}^{n} V(x_k) \subset V \circ V(G(\bar{y})) \subset U(G(\bar{y}))$ for every $\bar{y} \in B$. Thus G is $\mathscr{U}_\sigma^*$-l.H.s.c., hence (2.5) is satisfied. The assertion then follows immediately from Corollary 2.1. ∎

Remark 2.5. In the preceding proof we have seen that, if G is σ-u.s.c. (or σ-l.s.c.), then it is also $\mathscr{U}_\sigma^*$-u.H.s.c. (or $\mathscr{U}_\sigma^*$-l.H.s.c.); $\mathscr{U}_\sigma^*$ denotes the Stone–Čech uniformity. In fact, G is σ-l.s.c. if and only if it is $\mathscr{U}_\sigma^*$-l.H.s.c.: suppose that G, being $\mathscr{U}_\sigma^*$-l.H.s.c., is not σ-l.s.c. at some $y \in Y$. Then there is $x \in G(y)$ and a σ-open set A containing x such that, for every neighbourhood B of y, $G(\bar{y})$ is disjoint with A for some $\bar{y} \in B$. Since σ is completely regular, there is a σ-continuous function $\varphi: X \to [0, 1]$ such that $\varphi(x) = 1$ and $\varphi(X \backslash A) = 0$. For $V = \{(x_1, x_2); |\varphi(x_1) - \varphi(x_2)| \leq \frac{1}{2}\}$ we get obviously $V \in \mathscr{U}_\sigma^*$ and $x \notin V(G(\bar{y}))$, hence $G(y) \not\subset V(G(\bar{y}))$. It shows that G cannot be $\mathscr{U}_\sigma^*$-l.H.s.c. at y, which is the contradiction.

Moreover, if G is closed-valued (i.e., $G(y)$ is σ-closed for every $y \in Y$) and the topology σ is normal, then also the following equivalence holds: G is σ-u.s.c. if and only if G is $\mathscr{U}_\sigma^*$-u.H.s.c. To show it, we modify the reasoning of S. Dolecki and S. Rolewicz [6, Sect. 2]: suppose that G, being $\mathscr{U}_\sigma^*$-u.H.s.c., is not σ-u.s.c. at some $y \in Y$. Then there is a σ-neighbourhood A of $G(y)$ such that, for every neighbourhood B of y, $G(\bar{y}) \not\subset A$ for some $\bar{y} \in B$. Since $G(y)$ is σ-closed and σ is normal, there exists a σ-continuous function $\varphi: X \to [0, 1]$ such that $\varphi(G(y)) = 0$ and $\varphi(X \backslash A) = 1$. Then $V(G(y)) \subset A$ with $V = \{(x_1, x_2); |\varphi(x_1) - \varphi(x_2)| \leq \frac{1}{2}\} \in \mathscr{U}_\sigma^*$, hence $G(\bar{y}) \not\subset V(G(y))$, which shows that G cannot be $\mathscr{U}_\sigma^*$-u.H.s.c., the desired contradiction.

3. A General View to Extensions of the Problem

In the preceding section we have employed the precompact modification $\mathscr{U}^*$ of the original uniformity $\mathscr{U}$ to obtain some stable extension of the problem, whose solutions may be considered as generalized solutions of the original problem and which yields also some results for the original problem. The question appears naturally whether it is necessary to employ just the uniformity $\mathscr{U}^*$.

First we paraphrase some result from [9] which, however, deals with an unconstrained problem only; i.e., $G(y) = X$ for all $y \in Y$. We suppose that the function $f: X \times Y \to \bar{R}$ has got the property: the mapping $Y \to 2^{X \times \bar{R}}$ defined by $y \mapsto \operatorname{epi} f(\cdot, y) = \{(x, a) \in X \times \bar{R}; f(x, y) \leq a\}$ is continuous with respect to the topology $\tau((\mathscr{U} \times \mathscr{R})^{\mathrm{H}})$ on $2^{X \times \bar{R}}$. Taking some uniformity $\mathscr{V}$ on X, we may define the extended problem

$$\text{minimize } f^{\mathscr{V}}(x, y)$$

$$\text{subject to } x \in X_c^{\mathscr{V}},$$

where $X_c^{\mathscr{V}}$ is the set obtained by the completion of the uniform space $(X, \mathscr{V})$, and the function $f^{\mathscr{V}}: X_c^{\mathscr{V}} \times Y \to \bar{R}$ is defined by

$$f^{\mathscr{V}}(x, y) = \liminf_{\tilde{x} \to x,\, \tilde{x} \in X} f(\tilde{x}, y). \tag{3.1}$$

Obviously, when $\tau(\mathscr{U})$ is coarser than $\tau(\mathscr{V})$ and $f(\cdot, y): X \to \bar{R}$ is $\tau(\mathscr{U})$-l.s.c. in the usual sense (i.e., $\forall x \in X$ $\forall \varepsilon > 0$ $\exists U \in \mathscr{U}$ $\forall \tilde{x} \in U(x)$: $\varphi(f(\tilde{x}, y)) \geqslant \varphi(f(x, y)) - \varepsilon$ with $\varphi: \bar{R} \to [-1, 1]$ defined by, e.g., $\varphi(a) = a/(1 + |a|)$), then we have $f^{\mathscr{V}}(x, y) = f(x, y)$ for every $x \in X$, hence $f^{\mathscr{V}}$ is actually an extension of f (namely the so-called l.s.c.-regularization of f). We have relaxed the strong requirement of the extension of f by continuity like in (2.2) in order to be able to investigate a broader class of extensions of the problem. Besides, in the context of minimization it seems quite natural to treat l.s.c. functions instead of continuous ones (cf. also [9, 10]). We define naturally the set of "$\mathscr{V}$-generalized solutions" of P_y as $M^{\mathscr{V}}(y) = \{x \in X_c^{\mathscr{V}};\ f^{\mathscr{V}}(x, y) = \inf f^{\mathscr{V}}(X_c^{\mathscr{V}}, y)\}$. It is shown in [9] that under the above stated assumptions the multifunction $y \to M^{\mathscr{V}}(y)$ is $\tau(\mathscr{V}_c)$-u.s.c. provided $\mathscr{V}$ is coarser than $\mathscr{U}^*$. Conversely, if this multifunction is u.s.c. for every choice of Y and f with the stated property, then $\mathscr{V}$ is coarser than $\mathscr{U}^*$. We thus observe that in the unconstrained case there is a large amount of the uniformities that give stable extension of the problem. The uniformity $\mathscr{U}^*$ is the finest among them and the corresponding completion (or we may say compactification) $X_c^{\mathscr{U}^*}$ of X is the "largest" among them in the sense that, if $\mathscr{V} \subset \mathscr{U}^*$, there is a continuous surjection from $X_c^{\mathscr{U}^*}$ onto $X_c^{\mathscr{V}}$ fixing X. Moreover, $\inf f^{\mathscr{V}}(X_c^{\mathscr{V}}, y)$ is always equal to $m(y)$.

In our constrained case the situation is essentially different. Considering a uniformity $\mathscr{V}$ on X, we define the extended problem

$$P_y^{\mathscr{V}}: \text{minimize } f^{\mathscr{V}}(x, y)$$
$$\text{subject to } x \in G^{\mathscr{V}}(y),$$

where $f^{\mathscr{V}}$ is defined by (3.1) and $G^{\mathscr{V}}(y)$ is naturally the $\tau(\mathscr{V}_c)$-closure of $G(y)$ in $X_c^{\mathscr{V}}$. Of course, we define the marginal function $m^{\mathscr{V}}(y) = \inf f^{\mathscr{V}}(G^{\mathscr{V}}(y), y)$ and the marginal multifunction $M^{\mathscr{V}}(y) = \{x \in G^{\mathscr{V}}(y);\ f^{\mathscr{V}}(x, y) = m^{\mathscr{V}}(y)\}$; i.e., $m^{\mathscr{V}}(y)$ and $M^{\mathscr{V}}(y)$ are the infimum and the set of the minimizers of $P_y^{\mathscr{V}}$, respectively. It is obvious that, if $\mathscr{V} = \mathscr{U}^*$, then $X_c^{\mathscr{V}} = \bar{X}$ and $G^{\mathscr{V}} = \bar{G}$. If, in addition, f satisfies (2.1), then also $f^{\mathscr{V}} = \bar{f}$, $m^{\mathscr{V}} = \bar{m}$, and $M^{\mathscr{V}} = \bar{M}$ (in other words, $P_y^{\mathscr{V}}$ is nothing else than $\bar{P}_y$).

THEOREM 3.1. *Let $\mathscr{U}$ be the given uniformity and $\mathscr{V}$ be another uniformity on X such that the following implication holds*:

$$(2.1),\ (2.4),\ \textit{and}\ (2.5) \Rightarrow M^{\mathscr{V}}\ \textit{is}\ \tau(\mathscr{V}_c)\textit{-u.s.c.} \tag{3.2}$$

Then $\mathscr{V}$ is coarser than $\mathscr{U}^*$.

Proof. First we prove that, if (3.2) is valid, then $\mathscr{V}^*$ is coarser than $\mathscr{U}^*$. We will modify the arguments in the proof of Lemma 3 in [9]. Suppose the contrary, i.e., $\mathscr{V}^*$ is not coarser than $\mathscr{U}^*$ (in other words, the identity on X is not proximally continuous from the proximity induced by $\mathscr{U}$ to the proximity induced by $\mathscr{V}$). Then there is $A \subset X$ and $V \in \mathscr{V}^*$ such that $W(A) \not\subset V \circ V(A)$ for any $W \in \mathscr{U}$. Take $Y = \mathscr{U} \cup \{\Delta\}$ endowed with the topology defined by declaring $\{\{V \in Y; V \subset W\}\}_{W \in \mathscr{U}}$ as the base of the neighbourhood filter of the point $y = \Delta$ and $\{W\}$ as the base of the neighbourhood filter for $y = W \in \mathscr{U}$. Put $f = 0$ (thus (2.1) and (2.4) are satisfied trivially), and $G(W) = W(A)$ for every $W \in Y$; note that $G(\Delta) = A$. We thus get the situation: $G(W)$ converge to $G(\Delta)$ in the topology $\tau(\mathscr{U}^{\mathrm{H}})$ on 2^X when W ranges the filter $\mathscr{U}$ (thus (2.5) is satisfied), but $M^{\mathscr{V}}(W) = G^{\mathscr{V}}(W) = \mathrm{cl}_{X_c^{\mathscr{V}}} W(A) \not\subset \mathrm{cl}_{X_c^{\mathscr{V}}} V(A)$ for any $W \in \mathscr{U}$. As $\mathrm{cl}_{X_c^{\mathscr{V}}} V(A)$ is a $\tau(\mathscr{V}_c)$-neighbourhood of $\mathrm{cl}_{X_c^{\mathscr{V}}} A = M^{\mathscr{V}}(\Delta)$, we see that $M^{\mathscr{V}}$ is not u.s.c. at $y = \Delta$ and thus (3.1) is not valid, which is the contradiction.

It remains to prove that $\mathscr{V}$ must be precompact. We employ the (slightly modified) idea of the proof of Lemma 4 in [9]. Suppose that $\mathscr{V}$ is not precompact. Then there exists a $\mathscr{V}$-uniformly discrete sequence $\{x_n\}_{n \in N}$ in X; i.e., $\exists V \in \mathscr{V}$ (we may suppose V to be symmetric) $\forall n \neq m$: $V(x_n) \cap V(x_m) = \emptyset$. Furthermore, there is a $(\mathscr{V}, \mathscr{R})$-uniformly continuous function $\varphi: X \to [0, 1]$ such that $\varphi(x_n) = 0$ for every $n \in N$ and $\varphi(x) = 1$ for $x \in X \setminus \bigcup_{n \in N} V(x_n)$. Now take $Y = N \cup \{\infty\}$ endowed with the topology that $\{\{\bar{n} \in Y; \bar{n} \geqslant n\}\}_{n \in N}$ is a base of the neighbourhood filter of the point $y = \infty$, while every point from N is a neighbourhood of itself. Then put

$$f(x, \infty) = \begin{cases} (1 - 1/k)\,\varphi(x) + 1/k & \text{if } \; x \in V(x_k) \text{ for some } k \in N, \\ 1 & \text{elsewhere,} \end{cases}$$

and

$$f(x, n) = \begin{cases} \varphi(x) & \text{for } \; x \in V(x_n), \\ f(x, \infty) & \text{elsewhere.} \end{cases}$$

Obviously, $f(\cdot, y)$ is $(\mathscr{V}, \mathscr{R})$-uniformly continuous for every $y \in Y$, hence also $(\mathscr{V}^*, \mathscr{R})$-uniformly continuous. Since $\mathscr{V}^* \subset \mathscr{U}^* \subset \mathscr{U}$, $f(\cdot, y)$ is $(\mathscr{U}, \mathscr{R})$-uniformly continuous as well, i.e., (2.1) is valid. We have the estimate $0 \leqslant f(x, n) - f(x, \infty) \leqslant 1/n$, from which (2.4) clearly follows. Putting $G(y) = X$ for every $y \in Y$, (2.5) is satisfied trivially. Since $f(\cdot, \infty) \geqslant 0$ and $f(x_n, \infty) = 1/n$ for every $n \in N$, we see that $m(\infty) = m^{\mathscr{V}}(\infty) = 0$. Take sufficiently small, symmetric $W \in \mathscr{V}_c$ such that $(W \circ W) \cap (X \times X) \subset V$. Then, for every $x \in X_c^{\mathscr{V}}$, $W(x)$ is a $\tau(\mathscr{V}_c)$-neighbourhood of x such that there is at most one element x_m belonging to it because the set $\{x_n\}_{n \in N}$ is discrete. If $x_m \in W(x)$ for some $m \in N$, then also $x \in W(x_m)$, thus $W(x) \cap X \subset$

$W \circ W(x_m) \cap X \subset V(x_m)$, and we have got the estimate $f(\tilde{x}, \infty) \geqslant 1/m$ for all $\tilde{x} \in W(x) \cap X$, hence by (3.1), $f^{\mathscr{V}}(x, \infty) \geqslant 1/m$. If there does not exist any point x_m belonging to $W(x)$, we have even $f(\tilde{x}, \infty) = 1$ for every $\tilde{x} \in W(x) \cap X$, hence $f^{\mathscr{V}}(x, \infty) = 1$. In every case we have got $f^{\mathscr{V}}(x, \infty) > 0$ for all $x \in X_c^{\mathscr{V}}$, which shows that $M^{\mathscr{V}}(\infty)$ is empty. On the other hand, $M^{\mathscr{V}}(n)$ contains at least the point x_n, therefore the multifunction $M^{\mathscr{V}}$ is not u.s.c. at $y = \infty$, the contradiction. Thus we have shown that $\mathscr{V}$ must be precompact; and therefore $\mathscr{V} = \mathscr{V}^* \subset \mathscr{U}^*$. ∎

By Theorem 2.1 the implication (3.2) is valid for $\mathscr{V} = \mathscr{U}^*$. On the other hand, if $\mathscr{V} \neq \mathscr{U}^*$ (and $\mathscr{V} \subset \mathscr{U}^*$), both possibilities (i.e., (3.2) is true or not) may actually appear as shown in the following simple examples (note that $\mathscr{V}$ will always be a Hausdorff uniformity).

EXAMPLE 3.1. Take $X = [0, 1[\cup \{2\}$, $\mathscr{U}$ the trace on X of the standard additive uniformity $\mathscr{R}_0$ on R, and $\mathscr{V}$ the pre-image of $\mathscr{R}_0$ under the mapping $\xi: X \to [0, 1]$ defined by $\xi(a) = \min(a, 1)$, this means $V \in \mathscr{V}$ iff $\exists \varepsilon > 0$ $\forall x_1, x_2 \in X: |\xi(x_1) - \xi(x_2)| \leqslant \varepsilon \Rightarrow x_1 V x_2$. Note that $\mathscr{V}$ "sticks" the point 2 to the interval $[0, 1[$ from the right. Clearly, $\mathscr{V} \subset \mathscr{U}^* = \mathscr{U}$ because $\mathscr{U}$ is precompact and ξ is an $(\mathscr{R}_0, \mathscr{R}_0)$-uniformly continuous function. Considering $Y = [0, 1]$, $f(x, y) = \min(x, \frac{3}{2} - x)$, and $G(y) = [0, 1 - y]$, we get the situation that f fulfills (2.1) and (2.4), G fulfills (2.5), but $M^{\mathscr{V}}(1) = \{2\}$ while $M^{\mathscr{V}}(y) = \{0\}$ for every $y < 1$, hence $\mathscr{V}$ does not satisfy the implication (3.2).

EXAMPLE 3.2. Take $X = X_1 \cup X_2$, $X_1 = \{x \in [0, 1];\ x \text{ rational}\}$, $X_2 = \{x \in [2, 3];\ x \text{ irrational}\}$, $\mathscr{U}$ the trace on X of the standard additive uniformity $\mathscr{R}_0$ on R, and $\mathscr{V}$ the pre-image of $\mathscr{R}_0$ under the mapping $\xi: X \to [0, 1]$ defined by $\xi(a) = \min(a, 3 - a)$. Obviously, $\mathscr{V}$ "sticks" both "leafs" X_1 and X_2 to each other. Again we have $\mathscr{V} \subset \mathscr{U}^* = \mathscr{U}$ because $\mathscr{U}$ is precompact and ξ is $(\mathscr{R}_0, \mathscr{R}_0)$-uniformly continuous. Obviously, the space $(X, \mathscr{V})$ is homeomorphic to the compact interval $[0, 1]$. Identifying $X_c^{\mathscr{V}}$ with $[0, 1]$ and employing the density of both X_1 and X_2 in $X_c^{\mathscr{V}}$, we may write $f^{\mathscr{V}}(x, y) = \min(f^{\mathscr{U}}(x, y), f^{\mathscr{U}}(3 - x, y))$ and $G^{\mathscr{V}}(y) = G_1(y) \cup G_2(y)$, where $G_i = \mathrm{cl}\ \xi(G(y) \cap X_i)$, $i = 1, 2$, "cl" means the closure in $[0, 1]$. Due to (2.1) and (2.4), $f^{\mathscr{U}}$ is continuous, hence also $f^{\mathscr{V}}$ is obviously continuous. Due to (2.5), $G(y) \cap X_i$ is $\mathscr{R}_0$-l.H.s.c. and $\mathscr{R}_0$-u.H.s.c. for $i = 1, 2$, hence also G_i, $i = 1, 2$, have this property, and therefore $G^{\mathscr{V}}$ is $\mathscr{V}_c$-u.H.s.c. and $\mathscr{V}_c$-l.H.s.c. We see that $M^{\mathscr{V}}$ is then u.s.c. as a consequence of Proposition 1.1, thus $\mathscr{V}$ does satisfy the implication (3.2) while it differs from $\mathscr{U}^*$.

In the following theorems we impose on the extension some other natural requirements which extremely narrow the class of the extensions that fulfill them. Thus we want to show that the choice $\mathscr{V} = \mathscr{U}^*$ employed in Section 2 was not only fortuitous. This will also justify the extension proposed in Section 2. First we will deal with the requirement that f can be extended by continuity like in (2.2), which is undoubtedly the most natural way to extend f.

THEOREM 3.2. *Let the following implication hold*:

$$(2.1),\ (2.4),\ \textit{and}\ (2.5) \Rightarrow M^{\mathscr{V}}\ \textit{is}\ \tau(\mathscr{V}_c)-\textit{u.s.c. and}\ \text{“liminf”}\ \textit{in}\ (3.1)\ \textit{can be replaced by}\ \text{“lim”}.$$

Then $\mathscr{V} = \mathscr{U}^*$.

Proof. In view of Theorem 3.1 it remains to show $\mathscr{V} \supset \mathscr{U}^*$. Taking Y large enough and f so that $f(\cdot, y)$ ranges over all $(\mathscr{U}, \mathscr{R})$-uniformly continuous functions from X to $\bar{R}$ (when y ranges over Y), $\mathscr{U}^*$ is the coarsest uniformity that makes uniformly continuous $f(\cdot, y)$ for all $y \in Y$. If "lim" in (3.1) has a sense, then $f^{\mathscr{V}}(\cdot, y)$ is continuous for all $y \in Y$. Due to Theorem 3.1, $\mathscr{V}$ is coarser than $\mathscr{U}^*$, hence it is precompact. Thus $f^{\mathscr{V}}(\cdot, y)$, being continuous on a compact space $X_c^{\mathscr{V}}$, is also $(\mathscr{V}_c, \mathscr{R})$-uniformly continuous. Therefore $f(\cdot, y)$, being the trace on X of $f^{\mathscr{V}}(\cdot, y)$, is $(\mathscr{V}, \mathscr{R})$-uniformly continuous. We thus observe that $\mathscr{V}$ must be finer than $\mathscr{U}^*$. ∎

THEOREM 3.3. *Let the following implication hold*:

$$(2.1),\ (2.4),\ \textit{and}\ (2.5) \Rightarrow M^{\mathscr{V}}\ \textit{is}\ \tau(\mathscr{V}_c)\textit{-u.s.c. and}\ m^{\mathscr{V}} = m. \tag{3.3}$$

Then $\mathscr{V} = \mathscr{U}^*$.

Proof. Again $\mathscr{V} \subset \mathscr{U}^*$ because of Theorem 3.1. Suppose that $\mathscr{V} \neq \mathscr{U}^*$. The identity on X, being $(\mathscr{U}^*, \mathscr{V})$-uniformly continuous, can be extended by continuity to a $(\tau(\mathscr{U}_c^*), \tau(\mathscr{V}_c))$-continuous mapping $\Phi: X_c^{\mathscr{U}^*} \to X_c^{\mathscr{V}}$. We will prove that Φ cannot be injective. It is obvious when $\mathscr{V}$ is not a Hausdorff uniformity (recall that $\mathscr{U}$ is supposed as a Hausdorff uniformity). Hence suppose that $\mathscr{V}$ is a Hausdorff uniformity. Since both $\mathscr{V}$ and $\mathscr{U}^*$ are precompact, $X_c^{\mathscr{V}}$ and $X_c^{\mathscr{U}^*}$ are, in fact, homeomorphic to the so-called Smirnov compactifications of X regarding the respective proximities (for details see, e.g., [4. 7]), and we may use the arguments of the proof of the Smirnov theorem (see [4, Theorem 6.4.16]) to demonstrate that Φ is surjective, i.e., $\Phi(X_c^{\mathscr{U}^*}) = X_c^{\mathscr{V}}$. If Φ would also be injective, then it would be a homeomorphism because $\mathscr{U}^*$ is precompact and $\mathscr{V}$ is a Hausdorff uniformity; see [4, Theorem 5.3.13]. Due to the compactness of $X_c^{\mathscr{V}}$, Φ^{-1} would be even uniformly continuous, hence $\mathscr{V} = \mathscr{U}^*$. Since we have supposed $\mathscr{V} \neq \mathscr{U}^*$, Φ cannot be injective in any case, and there are

$x_0, x_1 \in X_c^{\mathscr{U}^*}$ such that $x_0 \neq x_1$ and $\Phi(x_0) = \Phi(x_1)$. In other words, the uniformity $\mathscr{V}$ "sticks" at least two points together. There exists a continuous function $\varphi: X_c^{\mathscr{U}^*} \to [0, 1]$ such that $\varphi(x_0) = 0$ and $\varphi(x_1) = 1$. Now we may put $G(y) = \{x \in X;\ \varphi(x) \geq \frac{1}{2}\}$ and $f(x, y) = \varphi(x)$ for every $x \in X$. Since $X_c^{\mathscr{U}^*}$ is compact, f fulfills (2.1). The conditions (2.4) and (2.5) are fulfilled trivially. Yet $m(y) = \frac{1}{2}$, while $m^{\mathscr{V}}(y) = 0$ because $f^{\mathscr{V}}(\Phi(x_0), y) = 0$ and $\Phi(x_0) = \Phi(x_1) \in G^{\mathscr{V}}(y)$. It shows that (3.3) is not valid unless $\mathscr{V} = \mathscr{U}^*$. ∎

Remark 3.1. The uniformity $\mathscr{U}^*$ is closely related with the so-called proximity structure induced by $\mathscr{U}$; see, e.g., [4, 7]. More precisely, there is a one-to-one monotone correspondence between all proximity structures and all precompact uniformities on X. The proximity-space theory, starting in general topology in the early 1950s with the works of V. A. Efremovič, has been systematically used for constrained optimization already in [11], where, however, the constraint (which are not formed by a set as in P_y, but by a mapping into some proximity space) has got a rather different nature related with the perturbational theory of duality. It may be concluded that Theorems 3.2 and 3.3 stress the role of a proximity structure in constrained optimization.

Remark 3.2. Note that the theorems in Section 3 always employ all possible data f and G satisfying (2.1), (2.4), and (2.5) for the given uniformity $\mathscr{U}$. When we are given by some fixed data f and G, there may exist a large amount of uniformities $\mathscr{U}$ for which f and G satisfy (2.1) and (2.5), and thus there may exist eventually a large amount of stable extensions of the concrete problem, i.e., the extensions with the properties: f can be extended by continuity as in (2.2), $\bar{m} = m$, and $\bar{M}$ is u.s.c.

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